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# **EIGENVECTOR ANALYSIS FOR MULTIPATH**

**Edmond Rusjan**

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# Eigenvector Analysis for Multipath

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### Final Report

Edmond Rusjan

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This research analyses the properties of singular vectors of circulant matrices. In particular, it proves that the singular values of circulant matrices are doubly degenerate. This results in a V pattern in the Fourier transform of the singular vectors. An additional structure in the Fourier space is observed and proven. These results provide a rigorous and firm foundation for the use of the Fourier space structures in the analysis of multipath.

## 1 Introduction and Background

This section introduces the notation and describes the necessary background to make the discussion in the rest of the report self contained and easier to follow.

### 1.1 Eigenvalue/Eigenvector Decomposition

Many problems in applied mathematics, science and engineering lead to the eigenvalue - eigenvector problem[Strang86]:

$$A\vec{x} = \lambda\vec{x} \tag{1}$$

where  $A$  is an  $n$  by  $n$  given matrix and where the unknowns are the eigenvalues  $\lambda$  and the (nontrivial) eigenvectors  $\vec{x}$ . Eq. (1) is nonlinear, because  $\lambda$  multiplies  $\vec{x}$ . If we can discover  $\lambda$ , then the equation for  $\vec{x}$  is linear[Strang88]:

$$(A - \lambda I)\vec{x} = \vec{0} \tag{2}$$

The key to the problem is to notice:

1. The vector  $\vec{x}$  is in the nullspace of  $A - \lambda I$

2. The number  $\lambda$  is chosen so that  $A - \lambda I$  has a (nontrivial) nullspace.

The eigenvalue - eigenvector problem of Eq. (1) is closely related to the problem of matrix diagonalization, also called the eigenvalue - eigenvector decomposition (EVD):

$$A = X\Lambda X^{-1} \quad (3)$$

where the diagonal matrix  $\Lambda = \text{diag}(\lambda_1, \dots, \lambda_n)$  is the eigenvalue matrix and the matrix  $X$  is the eigenvector matrix whose columns are the eigenvectors  $\vec{x}_1, \dots, \vec{x}_n$ .

### 1.1.1 EVD Existence

Not all matrices are diagonalizable. For example, the matrix:

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \quad (4)$$

is not diagonalizable. This is so because the eigenvalue  $\lambda = 0$  has algebraic multiplicity 2 and geometric multiplicity 1, i.e., it has only one independent eigenvector and we cannot construct  $X$ .

A matrix is diagonalizable iff it has  $n$  linearly independent eigenvectors, i.e., a complete set of eigenvectors. For example, any matrix with distinct eigenvalues is diagonalizable. Matrices who do not have a complete set of eigenvectors are called defective.

### 1.1.2 EVD (Non)uniqueness

The eigenvector matrix  $X$  is not unique, because any (nonzero) multiple of an eigenvector is again an eigenvector. Therefore any column of  $X$  can be multiplied by a (nonzero) constant and produce a new eigenvector matrix  $X$ . In addition, in the case of repeated eigenvalues, there is even more freedom. Any (nonzero) linear combination of eigenvectors corresponding to the repeated eigenvalue is again an eigenvector corresponding to that eigenvalue. Therefore such linear combinations of columns of  $X$  yield new eigenvector matrices  $X$ . In the extreme example, where  $A = I$ , the identity matrix, any invertible matrix  $X$  is an eigenvector matrix.

### 1.1.3 Eigenvalue/Eigenvector Ordering

The eigenvalues on the diagonal of the eigenvalue matrix  $\Lambda$  can be ordered arbitrarily as long as the corresponding eigenvectors appear in the same order as columns of the eigenvector matrix  $X$ . In the case of a repeated eigenvalue, the order of the corresponding eigenvectors is arbitrary.

### 1.1.4 Complex Eigenvalues/Eigenvectors

Eigenvalues and eigenvectors of real matrices may be complex. For example, consider the EVD of the rotation matrix:

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -i & i \end{bmatrix} \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -i & i \end{bmatrix}^{-1} \quad (5)$$

Symmetric matrices have real eigenvalues and eigenvectors, so their EVD involves real  $\Lambda$  and  $X$ .

### 1.1.5 Normal Matrices and Their Eigenvalues/Eigenvectors

A matrix  $N$  is normal if it commutes with its Hermitian (complex conjugate transpose), i.e., if  $NN^H = N^H N$ . Normal matrices are exactly those which possess a complete set of orthonormal eigenvectors. Therefore the EVD Eq.(3) has a special form for normal matrices:

$$N = U\Lambda U^H \quad (6)$$

where the matrix  $U$  is unitary and consequently  $U^{-1} = U^H$ . Examples of normal matrices are symmetric matrices, skew symmetric matrices, Hermitian and skew Hermitian matrices, orthogonal and unitary matrices. If  $N = S$  is symmetric, then  $U = Q$  is orthogonal and Eq.(6) becomes:

$$S = Q\Lambda Q^T \quad (7)$$

because  $Q^H = Q^T$ .

## 1.2 Singular Value/Vector Decomposition

Singular Value/Vector Decomposition (SVD) is the generalization of the EVD of a symmetric matrix Eq.(7) to cases where the EVD does not exist and in particular to rectangular matrices. Any  $m$  by  $n$  matrix  $A$  can be factored into:

$$A = Q_1 \Sigma Q_2^T = (\text{orthogonal})(\text{diagonal})(\text{orthogonal}) \quad (8)$$

where we have now two orthogonal matrices  $Q_1$  and  $Q_2$  (not necessarily transposes of each other) instead of one. The diagonal matrix  $\Sigma$  is called the singular value matrix and its entries  $\sigma_i$  are called the singular values. Singular values are nonnegative and  $\sigma_1, \sigma_2, \dots, \sigma_r$ , where  $r$  is the rank of  $A$  are positive.

The orthogonal matrices  $Q_1$  and  $Q_2$  are called the left and right singular vector matrices, respectively, and the columns are the (left and right) singular vectors.

The key to working with rectangular matrices  $A$  is, almost always, to consider  $A^T A$  and  $AA^T$ . The  $r$  nonzero singular values  $\sigma_1, \sigma_2, \dots, \sigma_r$  are the positive square

roots of the nonzero eigenvalues of both  $A^T A$  and  $AA^T$ . The columns of  $Q_1$  ( $m$  by  $m$ ) are eigenvectors of  $AA^T$  and the columns of  $Q_2$  ( $n$  by  $n$ ) are eigenvectors of  $A^T A$ .

For positive definite matrices, the SVD (Eq.8) is identical to the EVD (Eq.7). For indefinite matrices, any negative eigenvalues in  $\Lambda$  become positive in  $\Sigma$ . For example:

$$\begin{bmatrix} 2 & 0 \\ 0 & -3 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 3 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad (9)$$

For complex matrices  $\Sigma$  remains real, but  $Q_1$  and  $Q_2$  become unitary. Then (Eq.8) becomes:

$$A = U_1 \Sigma U_2^T = (\text{unitary})(\text{diagonal})(\text{unitary}) \quad (10)$$

The columns of  $Q_1$  and  $Q_2$  give orthonormal basis for all four fundamental subspaces:

	first	r	columns of $Q_1$ :	column space of A	(11)
	last	m-r	columns of $Q_1$ :	left nullspace of A	
	first	r	columns of $Q_2$ :	row space of A	
	last	n-r	columns of $Q_2$ :	nullspace of A	

The SVD chooses these bases in an extremely special way. They are more than just orthonormal. If  $A$  multiplies a column of  $Q_2$ , it produces a multiple of a column of  $Q_1$ . This comes directly from  $AQ_2 = Q_1 \Sigma$ , looked at a column at a time.

The connections with  $AA^T$  and  $A^T A$  and must hold if the formula  $Q_1 \Sigma Q_2^T$  is correct. That is easy to see:

$$AA^T = (Q_1 \Sigma Q_2^T)(Q_2 \Sigma^T Q_1^T) = Q_1 \Sigma \Sigma^T Q_1^T \quad \text{and similarly} \quad A^T A = Q_2 \Sigma^T \Sigma Q_2^T \quad (12)$$

and similarly .

From the first,  $Q_1$  must be the eigenvector matrix for  $AA^T$ . The eigenvalue matrix in the middle is  $\Sigma \Sigma^T$  - which is  $m$  by  $m$  with  $\sigma_1^2, \dots, \sigma_r^2$  on the diagonal. From the second,  $Q_2$  must be the eigenvector matrix for  $A^T A$ . The diagonal matrix  $\Sigma^T \Sigma$  has the same  $\sigma_1^2, \dots, \sigma_r^2$ , but it is  $n$  by  $n$ .

To stress the fact that  $A$  may be orthogonal, let us consider an example with a single column:

$$\begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix} = \begin{bmatrix} -\frac{1}{3} & \frac{2}{3} & \frac{2}{3} \\ \frac{2}{3} & -\frac{1}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{2}{3} & -\frac{1}{3} \end{bmatrix} \begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} 1 \end{bmatrix} \quad (13)$$

Here  $A^T A$  is 1 by 1, while  $AA^T$  is 3 by 3. They both have eigenvalue 9, whose square root is 3 in  $\Sigma$ . The two zero eigenvalues of  $AA^T$  leave some freedom for the eigenvectors in the second and third columns of  $Q_1$ . There are many choices which keep the matrix  $Q_1$  orthogonal.

### 1.2.1 SVD (Non)uniqueness

In general, the SVD is not unique. For example, consider  $A$  which is already orthogonal, i.e.,  $A = Q$ . Then the singular value matrix must be the identity matrix:  $\Sigma = I$ . This allows a lot of freedom for the decomposition. For example,  $A = QII$  or  $A = IIQ$  or even  $A = (QQ_2)IQ_2^T$ .

### 1.2.2 SVD Existence

We prove the existence by explicit construction. For arbitrary matrix  $A$  we need to find a diagonal matrix  $\Sigma$  and two orthogonal matrices  $Q_1$  and  $Q_2$  such that  $A = Q_1\Sigma Q_2^T$ .

Consider the  $n$  by  $n$  matrix  $A^T A$ . Since it is symmetric, it has a complete set of orthonormal eigenvectors  $\vec{x}_1, \dots, \vec{x}_n$  and corresponding eigenvalues  $\lambda_1, \dots, \lambda_n$ . Since  $A^T A$  is nonnegative definite, its  $r$  nonzero eigenvalues are positive and we can define:

$$\sigma_j = \sqrt{\lambda_j} \quad \text{and} \quad \vec{q}_j = \frac{A\vec{x}_j}{\sigma_j} \quad j = 1, \dots, r \quad (14)$$

Then, using Eq. (14):

$$Q_2 = [\vec{x}_1 | \vec{x}_2 | \dots | \vec{x}_n] \quad \Sigma = \text{diag}(\sigma_1, \sigma_2, \dots, \sigma_r) \quad Q_1 = [\vec{q}_1 | \vec{q}_2 | \dots | \vec{q}_m] \quad (15)$$

It is obvious that  $Q_2$  is orthogonal and that  $\Sigma$  is diagonal. We need to check that  $Q_1$  is orthogonal and that  $Q_1\Sigma Q_2 = A$ .  $Q_1$  is orthogonal because its columns  $\vec{q}_i$  are orthonormal:

$$\vec{q}_i^T \vec{q}_j = \frac{\vec{x}_i^T A^T A \vec{x}_j}{\sigma_i \sigma_j} = \frac{\lambda_i \vec{x}_i^T \vec{x}_j}{\sigma_i \sigma_j} = \delta_{ij} \quad (16)$$

Check that the SVD actually gives us  $A$ :

$$\vec{q}_i^T \vec{q}_j = \frac{\vec{x}_i^T A^T A \vec{x}_j}{\sigma_i \sigma_j} = \frac{\lambda_i \vec{x}_i^T \vec{x}_j}{\sigma_i \sigma_j} = \delta_{ij} \quad (17)$$

To check that  $Q_1\Sigma Q_2^T = A$  we note that the SVD can be written as a sum of rank 1 transformations:

$$Q_1\Sigma Q_2^T = \vec{q}_1\sigma_1\vec{x}_1 + \vec{q}_2\sigma_2\vec{x}_2 + \dots + \vec{q}_r\sigma_r\vec{x}_r \quad (18)$$

To  $Q_1\Sigma Q_2^T = A$  it is enough to check that both sides act the same way on basis vectors. For  $\vec{x}_j$  a basis vector in the row space:

$$Q_1\Sigma Q_2^T \vec{x}_j = (\vec{q}_1\sigma_1\vec{x}_1 + \vec{q}_2\sigma_2\vec{x}_2 + \dots + \vec{q}_r\sigma_r\vec{x}_r)\vec{x}_j = \vec{q}_j\sigma_j = \frac{A\vec{x}_j}{\sigma_j}\sigma_j = A\vec{x}_j \quad (19)$$

For a basis vector in the nullspace, both sides are zero. This completes the proof.

### 1.2.3 SVD Intuition

To summarize, the SVD takes the right singular vectors (eigenvectors of  $A^T A$ ), scales them by the singular values and "rotates them" into the left singular vectors:

$$A = Q_1 \Sigma Q_2^T = (\text{rotate})(\text{scale})(\text{project onto } A^T A \text{ eigenvectors}) \quad (20)$$

## 1.3 Circulant Matrices

An  $m$  by  $m$  matrix  $C$  is called circulant if each row of  $C$  can be obtained from the previous row by circular rotation of elements, i.e., if we shift each element in the  $i$ -th row over one column, with the element in the last column being shifted back to the first column, we get the  $(i+1)$ st row, unless  $i = m$ , in which case we get the first row. [Moon00] Thus the circulant matrix generated by  $(c_1, c_2, \dots, c_m)$  is:

$$C = \begin{bmatrix} c_1 & c_2 & c_3 & \dots & c_m \\ c_m & c_1 & c_2 & \dots & c_{m-1} \\ c_{m-1} & c_m & c_1 & \dots & c_{m-2} \\ \dots & & & & \\ c_2 & c_3 & c_4 & \dots & c_1 \end{bmatrix} \quad (21)$$

and is denoted for short  $C = \text{circ}(c_1, c_2, \dots, c_m) = \text{circ}(\vec{c})$  [Schott97].

Apart from the identity matrix  $I$ , the simplest circulant matrix is the permutation or the shift matrix:

$$\Pi = \text{circ}(0, 1, 0, \dots, 0) \quad (22)$$

Any circulant matrix  $C$  can be expressed as a polynomial in the shift matrix  $\Pi$ :

$$C = c_1 I + c_2 \Pi + \dots + c_m \Pi^{m-1} = p_{\vec{c}}(\Pi) \quad (23)$$

Every circulant matrix  $C$  is diagonalized by the discrete Fourier transform matrix  $F$ :

$$F = \begin{bmatrix} 1 & 1 & 1 & \dots & 1 \\ 1 & \omega & \omega^2 & \dots & \omega^{m-1} \\ 1 & \omega^2 & \omega^4 & \dots & \omega^{2(m-2)} \\ \dots & & & & \\ 1 & \omega^{m-1} & \omega^{2(m-1)} & \dots & \omega^{(m-1)(m-1)} \end{bmatrix} \quad (24)$$

where  $\omega = e^{\frac{-j2\pi}{m}}$ . This gives the EVD:

$$C = \frac{1}{n} F \Lambda F^H \quad (25)$$

with an explicit formula for the eigenvalues:

$$\Lambda = \text{diag}(p_{\vec{c}}(1), p_{\vec{c}}(\omega), \dots, p_{\vec{c}}(\omega^{m-1})) \quad (26)$$

We will find this formula very useful for calculating the EVD.

## 2 The $F^2$ Structure

In the analysis of multipath several structures have been observed in the Fourier space of the singular vectors of the data matrix[Larue04]. This opens two questions: what is the origin of these structures and how can they be used to improve the algorithms for the multipath detection.

In order to understand the more complicated structures which appear in realistic data, the simpler and generic data independent structures need to be understood first. We believe the simplest of these structures is what we call the  $F^2$  structure which we describe and explain now. It emerges when we study the eigenvectors of a generic circulant matrix  $C$  in the Fourier space.

The eigenvectors of a circulant matrix  $C$  are the columns of the Fourier matrix  $F$  (Eq.??). They are associated with a definite frequency. Suppose we want to analyze these eigenvectors in the Fourier space. That means taking their discrete Fourier transform, i.e., multiplying them by the DFT matrix, which is exactly the Fourier matrix  $F$ . In matrix form, this corresponds to  $FF = F^2$ .

We want to explain the structure which we observe by plotting the DFT of the eigenvectors, i.e., the  $F^2$ . The structure has a dot in the  $(0,0)$  corner and a line along the  $(0,n)$  to  $(n,0)$  diagonal.

By direct calculation:

$$(F^2)_{ij} = \sum_{k=0}^{n-1} \omega^{ik} \omega^{kj} = \sum_{k=0}^{n-1} (\omega^{i+j})^k = \frac{(\omega^{i+j})^n - 1}{\omega^{i+j} - 1} = \frac{(\omega^n)^{i+j} - 1}{\omega^{i+j} - 1} = 0 \quad (27)$$

as long as  $\omega^{i+j} \neq 1$ . Here we used the summation formula for the first  $n$  terms of a geometric series and the fact that  $\omega^n = 1$ . Therefore, the only nonzero elements of  $F^2$  are those where  $\omega^{i+j} = 1$ . This can happen in two cases:

1.  $i = j = 0$ : This explains the dot in the  $(0,0)$  corner of the matlab "imagesc" picture of the  $F^2$  matrix
2.  $i + j = n$ : This explains the straight line along the  $(0,n) - (n,0)$  diagonal of  $F^2$

In both cases:

$$\sum_{k=0}^{n-1} (\omega^{i+j})^k = \sum_{k=0}^{n-1} 1 = n \quad (28)$$

## 3 SVD of Circulant Matrices

This section illustrates intuitively, on an example, why the singular values of a circulant matrix come in pairs.

Let

$$C = \begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 1 & 2 \\ 2 & 0 & 0 & 1 \end{bmatrix} = I + 2\Pi \quad (29)$$

where  $\Pi$  is the permutation matrix:

$$\Pi = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix} \quad (30)$$

Then  $C^T = I + \Pi^T = I + \Pi^3$  and

$$C^T C = C C^T = 5I + 2S + 2S^3 = \begin{bmatrix} 5 & 2 & 0 & 2 \\ 2 & 5 & 2 & 0 \\ 0 & 2 & 5 & 2 \\ 2 & 0 & 2 & 5 \end{bmatrix} \quad (31)$$

To study the SVD  $C = Q_1 \Sigma Q_2^T$ , we look at  $C^T C$  and  $C C^T$ , because

$$C^T C = (Q_2 \Sigma^T Q_1^T)(Q_1 \Sigma Q_2^T) = Q_2 \Sigma^2 Q_2^T \quad (32)$$

$$C C^T = (Q_1 \Sigma Q_2^T)(Q_2 \Sigma^T Q_1^T) = Q_1 \Sigma^2 Q_1^T \quad (33)$$

and in general the singular values of  $C$  are the square roots of the eigenvalues of  $C^T C$  (or  $C C^T$ , they are the same), the singular vectors of  $C$  are the eigenvectors of  $C^T C$  and the left singular vectors of  $C$  are the eigenvectors of  $C C^T$ . For the special case of a circulant matrix, because of  $C^T C = C C^T$ , the singular vectors and the left singular vectors are the same, i.e.,  $Q_1 = Q_2$ .

So let us compute the eigenvalues of  $C^T C$  by Moon's theorem  $\lambda_i = p_{C^T C}(\omega^i) = 5 + 2\omega^i + 2\omega^{-i}$ , where  $\omega = e^{-\frac{j2\pi}{4}} = -j$ :

$$\lambda_0 = 5 + 2 + 2 = 9 \quad (34)$$

$$\lambda_1 = 5 - 2j + 2j = 5 \quad (35)$$

$$\lambda_2 = 5 - 2 - 2 = 1 \quad (36)$$

$$\lambda_3 = 5 + 2j - 2j = 5 \quad (37)$$

Note that  $\lambda_1 = \lambda_3$ . In general, it will be true that  $\lambda_i = \lambda_{n-i}$ , because the polynomial  $p_{C^T C}$  is symmetric under  $i \rightarrow n - i$ .

The  $C^T C$  is diagonalized by a Fourier matrix  $C^T C = \frac{1}{4} F \Lambda F^H$ , or explicitly:

$$C^T C = \begin{bmatrix} 5 & 2 & 0 & 2 \\ 2 & 5 & 2 & 0 \\ 0 & 2 & 5 & 2 \\ 2 & 0 & 2 & 5 \end{bmatrix} \quad (38)$$

$$= \frac{1}{4} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix} \begin{bmatrix} 9 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 5 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & j & -1 & -j \\ 1 & -1 & 1 & -1 \\ 1 & -j & -1 & j \end{bmatrix} \quad (39)$$

Note that  $C^T C$  has (at least) two diagonalizations, namely Eq. (33) and Eq. (39). Since diagonalization is essentially unique, i.e., up to a change of basis, the first (if we start counting with zero) and third column of  $Q_2$  are linear combinations of the first and third column of  $F$ . Since  $Q_2$  is real, we can simply take the real and imaginary part of the corresponding columns of  $F$ .

More explicitly, consider the spectral decomposition of  $C^T C$ :

$$C^T C = \frac{9}{4} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix} + \frac{1}{4} \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix} \begin{bmatrix} 1 & -1 & 1 & -1 \end{bmatrix} \quad (40)$$

$$+ \frac{5}{4} \left( \begin{bmatrix} 1 \\ -j \\ -1 \\ j \end{bmatrix} \begin{bmatrix} 1 & j & -1 & -j \end{bmatrix} + \begin{bmatrix} 1 \\ j \\ -1 \\ -j \end{bmatrix} \begin{bmatrix} 1 & -j & -1 & j \end{bmatrix} \right) \quad (41)$$

The two terms in parenthesis are complex conjugates of each other, so we can replace them with twice the real part of the first. In addition, rewrite the complex Fourier vector as a sum of two real waves:

$$\begin{bmatrix} 1 \\ -j \\ -1 \\ j \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix} - j \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{bmatrix} \quad (42)$$

This allows us to rewrite the parenthesis as

$$2\text{Re} \left( \begin{bmatrix} 1 \\ -j \\ -1 \\ j \end{bmatrix} \begin{bmatrix} 1 & j & -1 & -j \end{bmatrix} \right) \quad (43)$$

$$= 2 \left( \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & -1 & 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & -1 \end{bmatrix} \right) \quad (44)$$

Finally,  $C^T C$  can be rewritten as:

$$C^T C = \frac{9}{4} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} + \frac{1}{4} \begin{bmatrix} 1 & -1 & 1 & -1 \\ -1 & 1 & -1 & 1 \\ 1 & -1 & 1 & -1 \\ -1 & 1 & -1 & 1 \end{bmatrix} \quad (45)$$

$$+ \frac{5}{2} \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} + \frac{5}{2} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 5 & 2 & 0 & 2 \\ 2 & 5 & 2 & 0 \\ 0 & 2 & 5 & 2 \\ 2 & 0 & 2 & 5 \end{bmatrix} \quad (46)$$

which checks.

Singular values of  $C$  are square roots of the eigenvalues of  $C^T C$ , so:

$$\Sigma = \begin{bmatrix} 3 & 0 & 0 & 0 \\ 0 & \sqrt{5} & 0 & 0 \\ 0 & 0 & \sqrt{5} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad (47)$$

$Q_2$  is orthogonal, so need to make columns length 1:

$$Q_2 = \begin{bmatrix} \frac{1}{2} & \frac{1}{\sqrt{2}} & 0 & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{\sqrt{2}} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{1}{\sqrt{2}} & 0 & \frac{1}{2} \\ \frac{1}{2} & 0 & -\frac{1}{\sqrt{2}} & -\frac{1}{2} \end{bmatrix} \quad (48)$$

Note that we reordered the singular vectors as we are putting them into columns of  $Q_2$  in order to be aligned with the singular values, which are ordered from largest to smallest.

To find  $Q_1$ , compute  $CQ_2$ :

$$CQ_2 = \begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 1 & 2 \\ 2 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{2} & \frac{1}{\sqrt{2}} & 0 & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{\sqrt{2}} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{1}{\sqrt{2}} & 0 & \frac{1}{2} \\ \frac{1}{2} & 0 & -\frac{1}{\sqrt{2}} & -\frac{1}{2} \end{bmatrix} = \begin{bmatrix} \frac{3}{2} & \frac{1}{\sqrt{2}} & \frac{2}{\sqrt{2}} & -\frac{1}{2} \\ \frac{3}{2} & -\frac{2}{\sqrt{2}} & \frac{1}{\sqrt{2}} & \frac{1}{2} \\ \frac{3}{2} & -\frac{1}{\sqrt{2}} & -\frac{2}{\sqrt{2}} & -\frac{1}{2} \\ \frac{3}{2} & \frac{2}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & \frac{1}{2} \end{bmatrix} \quad (49)$$

and then divide the columns by the corresponding singular values to make them orthonormal, so

$$Q_1 = \begin{bmatrix} \frac{1}{2} & \frac{1}{\sqrt{10}} & \frac{2}{\sqrt{10}} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{2}{\sqrt{10}} & \frac{1}{\sqrt{10}} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{\sqrt{10}} & -\frac{2}{\sqrt{10}} & -\frac{1}{2} \\ \frac{1}{2} & \frac{2}{\sqrt{10}} & -\frac{1}{\sqrt{10}} & \frac{1}{2} \end{bmatrix} \quad (50)$$

Check:

$$C = \begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 1 & 2 \\ 2 & 0 & 0 & 1 \end{bmatrix} \quad (51)$$

$$= \begin{bmatrix} \frac{1}{2} & \frac{1}{\sqrt{10}} & \frac{2}{\sqrt{10}} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{2}{\sqrt{10}} & \frac{1}{\sqrt{10}} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{\sqrt{10}} & -\frac{2}{\sqrt{10}} & -\frac{1}{2} \\ \frac{1}{2} & \frac{2}{\sqrt{10}} & -\frac{1}{\sqrt{10}} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 3 & 0 & 0 & 0 \\ 0 & \sqrt{5} & 0 & 0 \\ 0 & 0 & \sqrt{5} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} & 0 \\ 0 & \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \end{bmatrix} \quad (52)$$

### 3.1 The V Conjecture

While one example is obviously not the proof, the hope is that we can prove that circulant matrices have singular values occuring in pairs, with the exception of  $\lambda_0$  and, for even  $n$ ,  $\lambda_{\frac{n}{2}}$ . The singular vectors are linear combinations of only two Fourier vectors. This is what gives two bright spots in the DFT of singular vectors. In cases where the Fourier coefficients decay monotonically with frequency, we get the "V" pattern.

### 3.2 The W Conjecture

In order to get the "W" pattern, which is empirically observed in many examples, we need more degeneracy, i.e., more singular values the same. We conjecture that the class of circulant matrices generated by a sinusoidal wave vector has the singular values sigmas (and therefore lambdas) come in groups of 4. Furthermore, the largest singular values correspond to the frequencies of the sinusoidal wave and the rest of the singular values decay monotonically with the distance from this frequency. This produces the structure shaped like the letter W.

## 4 Conclusion and Future Plans

This research provides additional evidence that the singular vector structures in the Fourier space, first observed and studied by James Larue[Larue04], are robust and relevant for the analysis of multipath. The existence of two of these structures is proven and their origin explained. Several more structures have been observed and we believe they deserve further study. In addition, the question of how to best use these structures to detect multipath is a promising direction for future research.

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